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Spin Symmetry for Dirac Equation with the Trigonometric Pöschl-Teller Potential

Xu-Yang Liu · Gao-Feng Wei · Xin-Wei Cao · Hong-Gang Bai

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Abstract Within the framework of the Dirac theory, the relativistic bound states for the trigonometric Pöschl-Teller (PT) potential are obtained in the case of spin symmetry. It is found from the numerical results that there exist only positive energy states for bound states in the case of spin symmetry. Also, the energy levels approach a constant when the potential parameter α goes to zero. The special case for equally scalar and vector trigonometric PT potential is also studied briefly.

Keywords Dirac equation · Trigonometric Pöschl-Teller potential · Spin symmetry

1 Introduction

The spin and pseudospin symmetry concepts introduced in nuclear theory [1, 2] are used to explain the features of deformed nuclei [3] and superdeformation [4], and to establish an effective shell-model coupling scheme [5]. Since the introduction of the spin and pseudospin symmetry concepts, some contribution to this problem has been made both in nuclear theory and in quantum theory [6–19]. For example, within the framework of the relativistic mean field theory, Ginocchio [6, 7] has found that a Dirac Hamiltonian with scalar and vector harmonic oscillator potentials in the case of $V(r) = S(r)$ possesses not only a spin symmetry but also a U(3) symmetry, but a Dirac Hamiltonian in the case of $V(r) = -S(r)$ possesses a pseudospin symmetry and a pseudo-U(3) symmetry. Meng et al. [8] have shown that the

X.-Y. Liu
School of Physics, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand
e-mail: lxy_gzu2005@126.com

G.-F. Wei (✉) · X.-W. Cao
Department of Physics, Xi'an University of Arts and Science, Xi'an 710065, P.R. China
e-mail: fgwei_2000@163.com

H.-G. Bai
Department of Basic Courses, Engineering College of Armed Police Force, Xi'an 710086, P.R. China

pseudospin symmetry is exact under the condition of $d(V(r)+S(r))/dr=0$ and spin symmetry is exact under the condition of $d(V(r)-S(r))/dr=0$. Also, under the condition of spin and pseudospin symmetry, some typical physical models have been studied such as harmonic oscillator [9, 10], Woods-Saxon potential [11], Morse potential [12–14], Eckart potential [15–17], Manning-Rosen potential [18] etc. In addition, Alhaidari et al. have investigated in detail physical interpretation on the three-dimensional Dirac equation in the case of spin symmetry limit $V(r)=S(r)$ and pseudospin symmetry limit $V(r)=-S(r)$ [19].

The trigonometric PT potential proposed for the first time by Pöschl and Teller [20] in 1933 was to describe the diatomic molecular vibration. Since the introduction this potential has received much attention and considerable interest. Although Chen [21] and Zhang et al. [22] have studied the relativistic bound state solutions for the trigonometric PT potential and hyperbolical PT (Second PT) potential in the case of equally vector and scalar potential case, respectively, the spin symmetry properties for the trigonometric PT potential has not been touched by far. In this work we attempt to study the spin symmetry properties for the trigonometric PT potential within the framework of the Dirac theory, which has not been performed to the best of our knowledge.

This paper is organized as follows. In Sect. 2 we derive the bound states of the Dirac equation with the trigonometric PT potential under condition of exact spin symmetry. In Sect. 3 we consider the special case for equally scalar and vector trigonometric PT potential. The conclusions are given finally in Sect. 4.

2 Spin symmetry

The Dirac equation of a nucleon with mass M moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ can be written as ($\hbar=c=1$)

$$\{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta[M+S(r)]\}\Psi(\boldsymbol{r})=[E-V(r)]\Psi(\boldsymbol{r}), \tag{1}$$

where E is the relativistic energy of the system, and α and β are the 4×4 Dirac matrices

$$\begin{aligned}\boldsymbol{\alpha}_i&=\begin{pmatrix}0&\boldsymbol{\sigma}_i\\ \boldsymbol{\sigma}_i&0\end{pmatrix},\qquad \beta=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\qquad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \sigma_2&=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\end{aligned} \tag{2}$$

To study the properties of spin symmetry, one can follow closely the procedure described in (1)–(5) of [23] to obtain

$$\left\{\frac{d^2}{dr^2}-\frac{k(k+1)}{r^2}-[M+E_{nk}-C]\sum(r)+[E_{nk}^2-M^2+C(M-E_{nk})]\right\}F_{nk}(r)=0, \tag{3}$$

where $\sum(r)=V(r)+S(r)$ is the sum potential, and $\Delta=V(r)-S(r)=C=\text{constant}$ is the difference potential. In this work we take the sum potential $\sum(r)$ as the trigonometric PT potential [21],

$$\sum(r)=\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}, \tag{4}$$

where parameters V_1 and V_2 describe the property of the potential well, $V_1>0$, $V_2>0$, while the parameter α is related to the range of this potential.

Obviously, (3) can be solved exactly only for s-wave ($k=-1$) case. Therefore, taking $k=-1$ and substituting (4) into (3) allows us to obtain

$$\left\{\frac{d^2}{dr^2}-(M+E_{n,-1}-C)\left[\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}\right] + [E_{n,-1}^2-M^2+C(M-E_{n,-1})]\right\}F_{n,-1}(r)=0. \tag{5}$$

Now we exactly solve (5) and study the spin symmetry properties for trigonometric PT potential. For this purpose, introducing a new variable $z=\sin^2(\alpha r)$, ($z\in(0,1)$) and inserting into (5), we have

$$\left\{z(1-z)\frac{d^2}{dz^2}+\left(\frac{1}{2}-z\right)\frac{d}{dz}-\frac{1}{4}\left[-\lambda^2+\frac{(M+E_{n,-1}-C)V_1}{\alpha^2 z} + \frac{(M+E_{n,-1}-C)V_1}{\alpha^2(1-z)}\right]\right\}F_{n,-1}(z)=0, \tag{6}$$

where

$$\lambda^2=\frac{E_{n,-1}^2-M^2+C(M-E_{n,-1})}{\alpha^2}. \tag{7}$$

Take trial wave function of the form

$$F_{n,-1}(z)=z^{\delta}(1-z)^{\mu}f_{n,-1}(z). \tag{8}$$

Inserting (8) into (6) yields

$$z(1-z)\frac{d^2f_{n,-1}(z)}{dz^2}+\left[\left(2\delta+\frac{1}{2}\right)-(2\delta+2\mu+1)z\right]\frac{df_{n,-1}(z)}{dz} -\left[(\delta+\mu)^2-\frac{1}{4}\lambda^2\right]f_{n,-1}(z)=0, \tag{9}$$

where

$$\begin{aligned}\delta&=\frac{1}{4\alpha}\left[\sqrt{4(M+E_{n,-1}-C)V_1+\alpha^2}+\alpha\right]>0,\\ \mu&=\frac{1}{4\alpha}\left[\sqrt{4(M+E_{n,-1}-C)V_2+\alpha^2}+\alpha\right]>0,\end{aligned} \tag{10}$$

whose hypergeometric function solution are given by [24]

$$f_{n,-1}(z)={}_2F_1(a,b;c;z), \tag{11}$$

with

$$a=-\lambda/2+\delta+\mu,\qquad b=\lambda/2+\delta+\mu,\qquad c=2\delta+1/2. \tag{12}$$

From the properties of hypergeometric functions, we know that series $f_{n,-1}(z)$ given in (11) approaches infinite unless parameter a is a negative integer. Therefore, considering the finiteness of solutions, the general quantum condition is given by

$$a=-\frac{1}{2}\lambda+\delta+\mu=-n,\quad (n=0,1,2,3,\ldots), \tag{13}$$

from which, together with (7) and (10), we obtain the energy equation as

$$E_{n,-1}^2 - M^2 + C(M - E_{n,-1}) = 4\alpha^2\left\{n + \frac{1}{2} + \frac{1}{4\alpha}\left[\sqrt{\alpha^2 + 4(M + E_{n,-1} - C)V_1} + \sqrt{\alpha^2 + 4(M + E_{n,-1} - C)V_2}\right]\right\}^2. \quad (14)$$

The energy level $E_{n,-1}$ is determined by energy equation (14), which is a rather complicated transcendental equation. The corresponding upper component $F_{n,-1}(r)$ can be expressed as

$$F_{n,-1}(r) = \sin^{2\delta}(\alpha r)\cos^{2\mu}(\alpha r)\,{}_2F_1\left(-n, n + 2\delta + 2\mu; 2\delta + \frac{1}{2}; \sin^2(\alpha r)\right). \quad (15)$$

With the help of the following recurrence relation of hypergeometric function

$$\frac{d}{dz}[{}_2F_1(a, b; c; z)] = \left(\frac{ab}{c}\right){}_2F_1(a + 1, b + 1; c + 1; z), \quad (16)$$

the corresponding lower component $G_{n,-1}(r)$ can be given by solving (4a) of [23] as follows

$$\begin{aligned} G_{n,-1}(r) &= \frac{1}{M + E_{n,-1} - C}\left[\frac{dF_{n,-1}(r)}{dr} - \frac{1}{r}F_{n,-1}(r)\right] \\ &= \frac{F_{n,-1}(r)}{M + E_{n,-1} - C}\left[2\alpha\delta\cot(\alpha r) - 2\alpha\mu\tan(\alpha r) - \frac{1}{r}\right] \\ &\quad - \frac{2\alpha n(n + 2\delta + 2\mu)}{(2\delta + 1/2)(M + E_{n,-1} - C)} \\ &\quad \times \sin^{2\delta+1}(\alpha r)\cos^{2\mu+1}(\alpha r)\,{}_2F_1\left(-n + 1, n + 2\delta + 2\mu + 1; 2\delta + \frac{3}{2}; \sin^2(\alpha r)\right). \end{aligned} \quad (17)$$

Thus by inserting $F_{n,-1}(r)$ and $G_{n,-1}(r)$ into (3) of [23], we can obtain the spinor wave functions of the Dirac equation for the trigonometric PT potential with spin symmetry. It is obvious that the upper and lower components satisfy the regularity boundary conditions when $\delta > 0, \mu > 0$, and $\delta, \mu \in \mathbb{R}$. If further considering the limit case $\alpha \to 0$, it is found from (14) that the energy eigenvalue approaches a constant, i.e. $\lim_{\alpha\to 0} E_{n,-1} = M + V_1 + V_2 \pm 2\sqrt{V_1 V_2}$ or $\lim_{\alpha\to 0} E_{n,-1} = C - M$, and the corresponding upper and lower components approach zero, i.e. $\lim_{\alpha\to 0} F_{n,-1}(r) = 0, \lim_{\alpha\to 0} G_{n,-1}(r) = 0$. That is to say, under the condition of spin symmetry there do not exist bound states in the limit case $\alpha \to 0$. However, the limit energy $\lim_{\alpha\to 0} E_{n,-1} = C - M$ is not physically acceptable since it makes the lower component $G_{n,-1}(r)$ diverge. In fact, the limit energy $\lim_{\alpha\to 0} E_{n,-1} = M + V_1 + V_2 - 2\sqrt{V_1 V_2}$ is also not realistic energy eigenvalue, which can be eliminated by numerical results of energy eigenvalues below.

The energy levels $E_{n,-1}$ are determined by energy (14), which is a rather complicated transcendental equation. As a numerical example, we take several sets of parameters to calculate the energy eigenvalues $E_{n,-1}$ and consider the effect of potential range parameter α on the energy levels as shown in Table 1. The numerical results show that the energy levels approach a constant of $\lim_{\alpha\to 0} E_{n,-1} = M + V_1 + V_2 + 2\sqrt{V_1 V_2}$ when potential range parameter α approaches zero, which is in good agreement with above theoretical analysis.

Table 1 The bound state energy levels $E_{n,-1}$ are shown in the case of spin symmetry. The numerical results show that the energy levels approach a constant when potential range parameter α approaches zero

n	$E_{n,-1}$					
	$C=2$, $V_1=5$, $V_2=3$, $M=10$					
	$\alpha=1.2$	$\alpha=0.8$	$\alpha=0.4$	$\alpha=0.2$	$\alpha=0.02$	$\alpha=0.002$
0	27.4300	26.8590	26.2975	26.0205	25.7733	25.7487
1	30.7157	29.0499	27.3923	26.5675	25.8280	25.7542
2	33.9942	31.2406	28.4881	27.1149	25.8826	25.7596
3	37.2570	33.4271	29.5841	27.6627	25.9373	25.7651
4	40.5003	35.6068	30.6797	28.2106	25.9920	25.7706
5	43.7224	37.7785	31.7744	28.7586	26.0466	25.7760
6	46.9232	39.9421	32.8679	29.3066	26.1013	25.7815

Moreover, it is found from the numerical results that there are only positive energy states for bound states in the case of spin symmetry.

3 Discussion

In this section we discuss above results (14), (15)–(17) in special case $\Delta(r)=C=0$. If so, we have $S(r)=V(r)=\frac{1}{2}\sum(r)$. Obviously, the obtained results will reduce to the bound states of the Dirac equation with equally scalar and vector potential case as follows,

$$E_{n,-1}^2-M^2=4\alpha^2\left\{n+\frac{1}{2}+\frac{1}{4\alpha}\left[\sqrt{\alpha^2+4(M+E_{n,-1})V_1}+\sqrt{\alpha^2+4(M+E_{n,-1})V_2}\right]\right\}^2. \quad (18)$$

The corresponding upper and lower components of Dirac spinors are expressed as

$$F_{n,-1}(r)=\sin^{2\delta}(\alpha r)\cos^{2\mu}(\alpha r)\,{}_2F_1\left(-n,n+2\delta+2\mu;2\delta+\frac{1}{2};\sin^2(\alpha r)\right), \quad (19)$$

$$\begin{aligned}G_{n,-1}(r)={}&\frac{F_{n,-1}(r)}{M+E_{n,-1}}\left[2\alpha\delta\cot(\alpha r)-2\alpha\mu\tan(\alpha r)-\frac{1}{r}\right]-\frac{2\alpha n(n+2\delta+2\mu)}{(2\delta+1/2)(M+E_{n,-1})}\\&\times\sin^{2\delta+1}(\alpha r)\cos^{2\mu+1}(\alpha r)\,{}_2F_1\left(-n+1,n+2\delta+2\mu+1;2\delta+\frac{3}{2};\sin^2(\alpha r)\right),\end{aligned} \quad (20)$$

where parameters $\delta$and μ are simplified as

$$\delta=\frac{1}{4\alpha}\left[\sqrt{4(M+E_{n,-1})V_1+\alpha^2}+\alpha\right],\qquad \mu=\frac{1}{4\alpha}\left[\sqrt{4(M+E_{n,-1})V_2+\alpha^2}+\alpha\right].$$

These above results (18)–(20) are the bound states of Dirac equation for equally scalar and vector trigonometric PT potentials.

4 Conclusions

In this work we have studied bound states of the Dirac equation for the trigonometric PT potential in the case of spin symmetry. The associated two-component spinors are obtained and the corresponding energy equation is derived. It is found from the numerical results that there are only positive energy states for bound states in the case of spin symmetry. Also, the energy levels approach a constant when potential range parameter α goes to zero. Finally, we consider the special case for equally scalar and vector potential case.

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